New algebra related to nonstandard R matrix



1993 J. Phys. A: Math. Gen. 26 L217

(http://iopscience.iop.org/0305-4470/26/5/007)

J. Phys. A: Math. Gen. 26 (1993) L217–L220. Printed in the UK

LETTER TO THE EDITOR

# New algebra related to non-standard $R$ matrix

Shao-Ming Fei and Rui-Hong Yue

CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China and (mailing address) Institute of Theoretical Physics, Academia Sinica, PO Box 2735, Beijing 100080, People's Republic of China

Received 27 October 1992

**Abstract.** A new algebra that is different to the known algebras in both quantum $R$ matrix and quantum enveloping algebras is presented. Its non-trivial Hopf algebraic structures are also discussed.

The quantum Yang–Baxter equation (QYBE) [1] has played important roles both in physics and mathematics. With solutions of QYBE one can construct exactly solvable models and find their eigenvalues and eigenstates [2]. On the other hand, any solution of QYBE can be generally used to find the new quasi-triangular Hopf algebra based on the method initiated by Faddeev *et al* [3].

Recently many multiparameter solutions ($4\times 4$ matrices) of QYBE have been obtained [4, 5]. Corresponding to the case of standard one-parameter $R$ matrix, the algebras related to the standard two-parameter $R$ matrix have also been discussed [6–8]. In this letter we investigate the group and algebraic structures related to one of the solutions in [4]. It is shown that the corresponding algebra is a new quantum algebra [9] with neither commutative nor co-commutative Hopf algebraic structures.

We consider the solution $R_2$ in [4]. After redefining the parameters we may write the matrix $R_2$ as

$$R=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&1+q&q&0\\0&0&0&1\end{pmatrix}. \tag{1}$$

The algebra related to this quantum matrix is governed by the Yang–Baxter equation [3]

$$RT_1T_2=T_2T_1R \tag{2}$$

where $T_1=T\otimes I$, $T_2=I\otimes T$, $I$ is $2\times 2$ identity matrix and

$$T=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

Equation (2) gives rise to the relations of the algebra elements $a$, $b$, $c$ and $d$

$$\begin{array}{lll}ab=q^{-1}ba & dc=qcd & bc=-qcb\\ bd=-db & ac=-ca & [a,d]=(1+q^{-1})bc\end{array} \tag{3}$$

where all the commutative relations have been omitted. This quantum matrix $T$ can be considered as a linear transformation of plane $\mathscr{A}_q(2)$ with coordinates $(x, \xi)$ satisfying $x\xi = -\xi x$. It is straightforward to prove that the coordinate transformations deduced by $T$

$$\begin{pmatrix} x' \\ \xi' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}$$

keep the relation $x'\xi' = -\xi' x'$. As there is no nilpotential element in quantum matrix $T$, there exists no constraints on the coordinates $x$ and $\xi$. This quantum plane is the same as the one related to $GL_q(2)$. However the matrix $R$ gives rise to a different kind of algebra.

From the algebraic relations (3) we can define an element of the algebra,

$$\delta(T) = \delta = ad - q^{-1}bc. \tag{4}$$

$\delta$ satisfies the following relations

$$\begin{aligned} &[a, \delta] = 0 \qquad [d, \delta] = 0 \\ &\{c, \delta\}_q \equiv qc\delta + \delta c = 0 \qquad \{\delta, b\}_q \equiv q\delta b + b\delta = 0. \end{aligned} \tag{5}$$

The element $\delta$ commutes only with $a$ and $b$ and hence is not the centre of the algebra. But it has the properties that can be proved directly,

$$\delta(TT') = \delta(T)\delta(T') \tag{6}$$

where $T$ and $T'$ are different solutions of equation (2) and the matrix elements of $T$ commutes with the ones of $T'$.

Now we formally define $\delta^{-1}$ to be the inverse of $\delta$. Then the inverse of $T$ can be written as

$$T^{-1} = \begin{pmatrix} d\delta^{-1} & -qb\delta^{-1} \\ -q^{-1}\delta^{-1}c & a\delta^{-1} \end{pmatrix}. \tag{7}$$

And the complete Hopf algebraic structures of $T$ are given by

$$\begin{aligned} &\Delta(a) = a \otimes a + b \otimes c \qquad \Delta(d) = d \otimes d + c \otimes b \\ &\Delta(b) = a \otimes b + b \otimes d \qquad \Delta(c) = d \otimes c + c \otimes a \\ &S(a) = d\delta^{-1} \qquad S(b) = -qb\delta^{-1} \\ &S(c) = -q^{-1}\delta^{-1}c \qquad S(d) = a\delta^{-1} \\ &\varepsilon(a) = \varepsilon(d) = 1 \qquad \varepsilon(b) = \varepsilon(c) = 0 \end{aligned} \tag{8}$$

where $\Delta$ is the coproduct, $S$ the antipode and $\varepsilon$ the co-unit. It can be proven that so defined operators $\Delta$ and $\varepsilon$ are algebraic homomorphism and $S$ is antihomomorphism. They satisfy all the necessary relations of Hopf algebra such as

$$\Delta(a)\Delta(d) - \Delta(d)\Delta(a) = (1 + q^{-1})\Delta(b)\Delta(c).$$

Hence the algebra given by (3) and (8) is a Hopf algebra. Moreover one can show that

$$\Delta(\delta) = \delta \otimes \delta \qquad S(\delta) = \delta^{-1} \qquad \varepsilon(\delta) = 1.$$

It is clear that $\delta$ has some properties of the quantum determinates in $SL_q(2)$ although it is not a centre here and this algebra is different to the usual form of $GL_q(2)$ [8].

Now we consider the quantum enveloping algebra related to the $R$ matrix (1). It is determined by the following Yang–Baxter relations [3],

$$RL_1^{\pm}L_2^{\pm}=L_2^{\pm}L_1^{\pm}R$$
$$RL_1^{+}L_2^{-}=L_2^{-}L_1^{+}R \tag{9}$$

where $L_1^{\pm}=L^{\pm}\otimes I$, $L_2^{\pm}=I\otimes L^{\pm}$, $L^+$ and $L^-$ are up-triangular and down-triangular matrices respectively,

$$L^+=\begin{pmatrix}k & x\\ 0 & l\end{pmatrix} \qquad L^-=\begin{pmatrix}m & 0\\ y & n\end{pmatrix}.$$

From (1) and (9) we have the following non-commutative relations

$$\begin{array}{lll} kx=qxk & mx=-xm & lx=-xl \\ nx=qxn & ky=q^{-1}yk & my=-ym \\ ly=-yl & ny=q^{-1}yn & qxy+yx=(1+q)(nk-ml). \end{array} \tag{10}$$

The algebra elements in (10) are not independent. After a survey of the relations we assume that $k=n$, $l=n$, $x^+=k^{-1/2}x$, $x^-=k^{-1/2}y$. Then the algebraic relations reduce to

$$\begin{array}{ll} kx^{\pm}=q^{\pm1}x^{\pm}k & lx^{\pm}=-x^{\pm}l \\ [k,l]=0 & \{x^+,x^-\}=\dfrac{1+q}{\sqrt{q}}(k-k^{-1}l^2). \end{array} \tag{11}$$

Algebra (11) may be re-expressed in a more elegant form

$$[H,E^{\pm}]=\pm E^{\pm} \qquad \{E^+,E^-\}=\frac{q^H-q^{-H}(-1)^{2H}}{q^{1/2}-q^{-1/2}} \tag{12}$$

where we have used the transformations

$$l=(-1)^H \qquad k=q^H \qquad E^{\pm}=(q-q^{-1})^{-1/2}x^{\pm}.$$

The corresponding Hopf structures of algebra (12) can be defined by

$$\begin{aligned} &\Delta(H)=H\otimes I+I\otimes H \\ &\Delta(E^+)=q^{H/2}\otimes E^++E^+\otimes q^{-H/2}(-1)^H \\ &\Delta(E^-)=q^{H/2}\otimes E^-+E^-\otimes q^{-H/2}(-1)^H \\ &\varepsilon(H)=\varepsilon(E^{\pm})=0 \qquad \varepsilon(\mathbf{1})=1 \\ &S(H)=-H \qquad S(E^{\pm})=(-1)^{\mp H}q^{\mp1/2}E^{\pm}. \end{aligned} \tag{13}$$

It is easy to show that $\Delta$ and $\varepsilon$ are algebraic homomorphism and $S$ is antihomomorphism. They satisfy three axioms of Hopf algebra

$$\begin{aligned} &(id\otimes\Delta)\Delta=(\Delta\otimes id)\Delta \\ &(id\otimes\varepsilon)\Delta(a)=(\varepsilon\otimes id)\Delta(a)=a \\ &m(S\otimes id)\Delta(a)=m(id\otimes S)\Delta(a)=\varepsilon(a)I \end{aligned}$$

where $a$ is any element of algebra (12) and $m$ is the multiplication operator.

So far we have presented the whole algebra structures, the algebra of quantum matrix and the quantum enveloping algebra, related to the solution (1) of QYBE. With neither commutative nor co-commutative Hopf algebraic structures, this algebra is a new quantum algebra. Here the parameter $q$ plays the role of 'quantization' of the algebra. When $q$ approaches one the algebra (10) becomes

$$\{l, x\}=0 \qquad \{l, y\}=0 \qquad \{x, y\}=2((-1)^{2H}-q^{2H}) \tag{14}$$

by setting $l=(-1)^H$ and $k=q^H$. Algebra (14) is still a 'quantum algebra' with neither commutative nor co-commutative Hopf algebraic structures. However when $q$ approaches $-1$, algebra (10) simply becomes

$$\{l, x\}=0 \qquad \{l, y\}=0 \qquad \{x, y\}=0.$$

The related Hopf algebraic structure becomes trivial.

In addition we would like to indicate that related to the solution (1) there is also a Temperley–Lieb algebraic representation. Let $\check{R}=R\cdot P$. $P$ is the permutation matrix. $\check{R}$ has two different eigenvalues 1 and $q$ satisfying

$$(\check{R}-1)^3(\check{R}-q)=(\check{R}-1)(\check{R}-q)=0.$$

Therefore a Temperley–Lieb algebraic representation can be readily given with element

$$e_i=1^{(1)}\otimes 1^{(2)}\otimes\cdots\otimes 1^{(i-1)}\otimes E\otimes 1^{(i+2)}\otimes\cdots\otimes 1^{(N)} \tag{15}$$

where $1^{(i)}$ represents a $2\times 2$ identity matrix at site $i$, $\otimes$ denotes the usual matrix tensor and

$$E=\sqrt{\frac{-1}{q}}(\check{R}-1).$$

$e_i$ satisfies the TL algebraic relations [10],

$$e_i^2=\frac{1-q}{\sqrt{-q}}e_i \qquad e_ie_{i\pm1}e_i=e_i$$
$$e_ie_j=e_je_i \qquad \text{if } |i-j|\geqslant 2. \tag{16}$$

Supported in part by the National Natural Science Foundation of China and LWTZ-1298 of Chinese Academy of Sciences.

## References

[1] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312–4

[2] de Vega H J and Woynarovich F LPTHE-91-35

Baxter R J 1982 *J. Stat. Phys.* **28** 1 and 1982 *Exactly Solved Models in Statistical Mechanics* (NewYork: Academic Press)

[3] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1990 *Leningrad Math. J.* 193–225

[4] Shao-Ming Fei, Han-Ying Guo and He Shi 1992 *J. Phys. A: Math. Gen.* **25** 2711–20

[5] Hietarinta J TURKU-FL-R7

[6] Schirrmacher A, Wess J and Zumino B 1991 *Z. Phys.* C **49** 317

[7] Burdik C and Hlavaty L 1991 *J. Phys. A: Math. Gen.* **24** L165

[8] Aghamohammadi A and Karimipour V SUTDP-92-70-4

[9] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247; 1986 *Commun. Math. Phys.* **102** 537

Faddeev L D 1982 *Les Houches Lectures*

Kulish P P and Sklyanin E K *Lecture Notes in Physics* vol 151, 61

[10] Temperley H N V and Lieb E 1971 *Proc. R. Soc.* A **322** 251